MAT 210 Filters

Recall: Our original goal developing convolution was to change Fourier coefficients $\mathcal{F}\{f\}$ **without** computing $\mathcal{F}$ & $\mathcal{F}^{-1}$.

$$\mathcal{F}_k\{f\circledast g\} = N\,\mathcal{F}_k\{f\}\cdot\mathcal{F}_k\{g\}$$

To change $\mathcal{F}\{f\} = [c_0\ c_1\ c_2\ c_3\ c_4\ c_5\ c_6\ c_7]$

$$\Downarrow$$

$$\mathcal{F}\{f\circledast g\} = [0\ 0\ c_2\ c_3\ 0\ c_5\ c_6\ 0]$$

$$\shortparallel$$

$$[0c_0\ \ 0c_1\ \ 1c_2\ \ 1c_3\ \ 0c_4\ \ 1c_5\ \ 1c_6\ \ 0c_7]$$

Use $\mathcal{F}\{g\} = \frac{1}{N}[0\ 0\ 1\ 1\ 0\ 1\ 1\ 0]$

(Note: $N=8$)

$$g = \frac{1}{8}\mathcal{F}^{-1}\{[0\ 0\ 1\ 1\ 0\ 1\ 1\ 0]\}$$

⋮

(Short IFFT calculation)

⋮

$$g = \tfrac{1}{8}\left[4\ \ -\sqrt{2}\ \ -2\ \ \sqrt{2}\ \ 0\ \ \sqrt{2}\ \ -2\ \ -\sqrt{2}\right]$$

Ex: Find g so that

$$\mathcal{F}\{f\circledast g\} = [0\ \ c_1\ \ 2c_2\ \ c_3]$$

$$\shortparallel$$

$$[0c_0\ \ 1c_1\ \ 2c_2\ \ 1c_3]$$

Use $\mathcal{F}\{g\} = \tfrac{1}{4}\cdot[0\ 1\ 2\ 1]$

$$g = \tfrac{1}{4}\mathcal{F}^{-1}\{[0\ 1\ 2\ 1]\}$$

$$= \tfrac{1}{4}[4\ \ -2\ \ 0\ \ -2]$$

$(0+1+2+1)$ $(0+i-2-i)$ ...

Check: Let $f = [1\ 2\ 3\ 4]$

$\hookrightarrow \mathcal{F}\{f\} = \tfrac{1}{4}[10\ \ -2+2i\ \ -2\ \ -2-2i]$

$$f\circledast g = \tfrac{1}{4}[-8\ 0\ 0\ 8]$$

$$= [-2\ 0\ 0\ 2]$$

$$\begin{array}{rrrr}
4 & 3 & 2 & 1 \\
-2 & 0 & -2 & 4 \\
\hline
16 & 12 & 8 & 4 \\
-6 & -4 & -2 & -8 \\
0 & 0 & 0 & 0 \\
-2 & -8 & -6 & -4 \\
\hline
8 & 0 & 0 & -8
\end{array}$$

$$\mathcal{F}\{f\circledast g\} = \tfrac{1}{4}[0\ \ -2+2i\ \ -4\ \ -2-2i]$$

$$\shortparallel$$

$$\tfrac{1}{4}[0\cdot 10\ \ 1(-2+2i)\ \ 2(-2)\ \ 1(-2-2i)]$$

Note: For long vectors it may actually be more work to compute $\underline{f} \circledast \underline{g}$ than to do $\mathcal{F}$ & $\mathcal{F}^{-1}$.

In order for $\underline{f} \circledast \underline{g}$ to be easy to compute we usually want $\underline{g}$ to be mostly 0. Finding a good $\underline{g}$ which is almost all 0 but also basically changes $\mathcal{F}_k\{\underline{f}\}$ the way you want is called "filter design".

Inverse Filters

Computations are slightly easier in the other direction.

$$\mathcal{F}_n^{-1}\{c \circledast d\} = \mathcal{F}_n^{-1}\{c\} \cdot \mathcal{F}_n^{-1}\{d\}$$

Ex: Find d so that

$$\mathcal{F}^{-1}\{c \circledast d\} = [0 \quad f_1 \quad 0 \quad 0]$$

$$= [0 f_0 \quad 1 f_1 \quad 0 f_2 \quad 0 f_3]$$

Use $\mathcal{F}^{-1}\{d\} = [0 \quad 1 \quad 0 \quad 0]$

$$d = \mathcal{F}\{[0 \quad 1 \quad 0 \quad 0]\}$$

$$= \tfrac{1}{4}[1 \quad -i \quad -1 \quad i]$$

(A)

The discrete convolution theorem (proof)

Recall: $\mathcal{F}_k\{f\} = \frac{1}{N}\sum_{n=0}^{N-1} f_n (\bar{\omega}_N^{\,k})^n$

and

$$(f \circledast g)_n = \sum_{s+t\equiv n} f_s \cdot g_t$$

$$\mathcal{F}_k\{f\circledast g\} = \frac{1}{N}\sum_{n=0}^{N-1} (f\circledast g)_n (\bar{\omega}_N^{\,k})^n$$

$$= \frac{1}{N}\sum_{n=0}^{N-1}\Big(\sum_{s+t\equiv n} f_s\cdot g_t\Big)(\bar{\omega}_N^{\,k})^n$$

$s+t\equiv n$ so $\bar{\omega}_N^{\,n} = \bar{\omega}_N^{\,s+t} = \bar{\omega}_N^{\,s}\,\bar{\omega}_N^{\,t}$

$$= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{s+t\equiv n} f_s(\bar{\omega}_N^{\,k})^s \cdot g_t(\bar{\omega}_N^{\,k})^t$$

sum over $s+t\equiv n$ for all n $\Updownarrow$ sum over all s, t

$$= \frac{1}{N}\sum_{s=0}^{N-1}\sum_{t=0}^{N-1} f_s(\bar{\omega}_N^{\,k})^s \cdot g_t(\bar{\omega}_N^{\,k})^t$$

$$= \frac{1}{N}\sum_{s=0}^{N-1} f_s(\bar{\omega}_N^{\,k})^s \cdot \sum_{t=0}^{N-1} g_t(\bar{\omega}_N^{\,k})^t$$

$$= \mathcal{F}_k\{f\}\cdot N\,\mathcal{F}_k\{g\}$$

$$\mathcal{F}_k\{f\circledast g\} = N\,\mathcal{F}_k\{f\}\cdot\mathcal{F}_k\{g\}$$

Inverse (proof)

Recall $\mathcal{F}_n^{-1}\{c\} = \sum_{k=0}^{N-1} c_k(\omega_N^{\,n})^k$

$$\mathcal{F}_n^{-1}\{c\circledast d\} = \sum_{k=0}^{N-1}(c\circledast d)_k(\omega_N^{\,n})^k$$

$$= \sum_{k=0}^{N-1}\Big(\sum_{s+t\equiv k} c_s\cdot d_t\Big)(\omega_N^{\,n})^k$$

$$= \sum_{k=0}^{N-1}\sum_{s+t\equiv k} c_s(\omega_N^{\,n})^s \cdot d_t(\omega_N^{\,n})^t$$

$$= \sum_{s=0}^{N-1}\sum_{t=0}^{N-1} c_s(\omega_N^{\,n})^s \cdot d_t(\omega_N^{\,n})^t$$

$$= \sum_{s=0}^{N-1} c_s(\omega_N^{\,n})^s \cdot \sum_{t=0}^{N-1} d_t(\omega_N^{\,n})^t$$

$$\mathcal{F}_n^{-1}\{c\circledast d\} = \mathcal{F}_n^{-1}\{c\}\cdot\mathcal{F}_n^{-1}\{d\}$$